Journal of Combinatorial Theory, Series A 217 (2026) 106093

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

journal homepage: www.elsevier.com/locate/jcta

Stirling permutation codes. II

Shi-Mei Ma [a,*], Hao Qi [b], Jean Yeh [c,*], Yeong-Nan Yeh [b]

[a] School of Mathematics and Statistics, Shandong University of Technology, Zibo 255000, Shandong, China
[b] College of Mathematics and Physics, Wenzhou University, Wenzhou 325035, China
[c] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan

ARTICLE INFO

Article history:
Received 8 June 2024
Received in revised form 3 June 2025
Accepted 25 June 2025
Available online xxxx

Keywords:
Eulerian polynomials
Stirling permutations
Signed permutations
e-Positivity

ABSTRACT

In the context of Stirling polynomials, Gessel and Stanley introduced Stirling permutations, which have attracted extensive attention over the past decades. Recently, we introduced Stirling permutation codes and provided numerous equidistribution results as applications. The purpose of the present work is to further analyze Stirling permutation codes. First, we derive an expansion formula expressing the joint distribution of the types A and B descent statistics over the hyperoctahedral group, and we also find an interlacing property involving the zeros of its coefficient polynomials. Next, we prove a strong connection between signed permutations in the hyperoctahedral group and Stirling permutations. We also study unified generalizations of the trivariate second-order Eulerian and ascent-plateau polynomials. Using Stirling permutation codes, we provide expansion formulas for eight-variable and seventeen-variable polynomials, which imply several e-positive expansions and clarify the connection among several statistics. Our results generalize the results of Bóna, Chen-Fu, Dumont, Haglund-Visontai, Janson and Petersen.

* Corresponding authors.
E-mail addresses: shimeimapapers@163.com (S.-M. Ma), qihao@wzu.edu.cn (H. Qi), chunchenyeh@mail.nknu.edu.tw (J. Yeh), mayeh@math.sinica.edu.tw (Y.-N. Yeh).

https://doi.org/10.1016/j.jcta.2025.106093
0097-3165/

Contents

1. Introduction

Let $[n]=\{1,2,\ldots,n\}$ and let $\pm[n]=[n]\cup\{\overline{1},\overline{2},\ldots,\overline{n}\}$, where $\overline{i}=-i$. The *symmetric group* $\mathfrak{S}_n$ is the group of all permutations on $[n]$, and the *hyperoctahedral group* $\mathfrak{S}_n^B$ is the group of signed permutations on $\pm[n]$ with the property that $\pi\left(\overline{i}\right)=-\pi(i)$ for all $i\in[n]$. The classical *Eulerian polynomials* over the symmetric group $\mathfrak{S}_n$ are defined by

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}(\pi)},$$

where $\operatorname{des}(\pi)$ is the number of descents of π, i.e., $\operatorname{des}(\pi)=\#\{i\in[n-1]\mid\pi(i)>\pi(i+1)\}$.

Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n^B$, and let $\operatorname{neg}(\pi)$ be the number of *negative entries* of π. The numbers of *types A and B descent* statistics of π are respectively defined by

$$\operatorname{des}_A(\pi)=\#\{i\in\{1,\ldots,n-1\}\mid\pi(i)>\pi(i+1)\},$$
$$\operatorname{des}_B(\pi)=\#\{i\in\{0,1,\ldots,n-1\}\mid\pi(i)>\pi(i+1),\ \pi(0)=0\}.$$

Thus $\operatorname{des}_B(\pi)=\operatorname{des}_A(\pi)$ if $\pi(1)>0$ and $\operatorname{des}_B(\pi)=\operatorname{des}_A(\pi)+1$ if $\pi(1)<0$. In [4], Brenti studied the following *Eulerian polynomials of type B* and their q-analogs:

$$B_n(x)=\sum_{\pi\in\mathfrak{S}_n^B}x^{\operatorname{des}_B(\pi)},\ B_n(x,q)=\sum_{\pi\in\mathfrak{S}_n^B}x^{\operatorname{des}_B(\pi)}q^{\operatorname{neg}(\pi)}.$$

Clearly, $B_n(x,0)=A_n(x)$. Since then, there has been a growing interest in the similar properties of Eulerian-type polynomials, including unimodality, real-rootedness, γ-positivity as well as algebraic and geometric interpretations, see [18,24,29,34,37,41].

A remarkable result of Foata-Schützenberger [15] says that

$$A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_1(n,k) x^k (1+x)^{n-1-2k},$$

where $\gamma_1(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k descents and without double descents. For $\pi \in \mathfrak{S}_n$, a *double descent* of π is an index $i \in [n-1]$ such that $\pi(i-1) > \pi(i) > \pi(i+1)$, where we set $\pi(0) = +\infty$; an *interior peak* of π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$; a *left peak* of π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we set $\pi(0) = 0$. Let $\gamma_2(n,k)$ (resp. $\gamma_3(n,k)$) be the number of permutations in $\mathfrak{S}_n$ with k interior peaks (resp. left peaks). By introducing modified Foata-Strehl action, Brändén [3] deduced that

$$A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{n-1-2k}} \gamma_2(n,k) x^k (1+x)^{n-1-2k}. \tag{1}$$

Using the theory of enriched P-partitions, Petersen [33, Proposition 4.15] obtained the following γ-positive expansion:

$$B_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} 4^i \gamma_3(n,i) x^i (1+x)^{n-2i}. \tag{2}$$

Consider the polynomials

$$b_n(x,y) = \sum_{\pi \in \mathfrak{S}_n^B} x^{\mathrm{des}_A(\pi)} y^{\mathrm{des}_B(\pi)}.$$

Clearly, $b_n(x,1) = 2^n A_n(x)$ and $b_n(1,x) = B_n(x)$. According to [1], the number of *flag descents* of $\pi \in \mathfrak{S}_n^B$ equals $\mathrm{des}_A(\pi) + \mathrm{des}_B(\pi)$. Hence $b_n(x,x)$ reduces to the flag descent polynomial. The *up-down runs* of $\pi \in \mathfrak{S}_n$ are the maximal consecutive subsequences that are increasing or decreasing of π endowed with a 0 in the front (see [31,38,40, 41] for details). Let $\mathrm{udrun}(\pi)$ denote the number of up-down runs of π. For example, $\mathrm{udrun}(623415) = \mathrm{udrun}(0623415) = 5$. Next present a unified extension of (1) and (2).

Theorem 1. *For any $n \geqslant 2$, the bivariate polynomial $b_n(x,y)$ has the expansion formula:*

$$b_n(x,y) = (1+y) \sum_{k \geqslant 0} 4^k \xi(n,k) (xy)^k (1+xy)^{n-1-2k} + y(1+x) \sum_{\ell \geqslant 0} 4^k \zeta(n,k) (xy)^k (1+xy)^{n-2-2k},$$

where $\xi(n,k) = T(n, 2k+1)$, $\zeta(n,k) = 2T(n, 2k+2)$, and $T(n,k)$ is the number of permutations in the symmetric group $\mathfrak{S}_n$ with k up-down runs.

As illustrations of Theorem 1, we have $b_1(x,y)=1+y$, $b_2(x,y)=(1+y+xy+xy^2)+(2y+2xy)$ and $b_3(x,y)=(1+y+10xy+10xy^2+x^2y^2+x^2y^3)+(6y+6xy+6xy^2+6x^2y^2)$. In Theorem 7, we shall establish a strong connection between the joint distribution of $(\mathrm{des}_A,\mathrm{des}_B,\mathrm{neg}\,)$ over $\mathfrak{S}_n^B$ and the joint distribution of $(\mathrm{lap}\,,\mathrm{ap}\,,\mathrm{even}\,)$ over restricted Stirling permutations.

The study of Stirling permutations originated from the work of Ramanujan [35], when he considered the Taylor series expansion:

$$\mathrm{e}^{nx}=\sum_{r=0}^{n}\frac{(nx)^r}{r!}+\frac{(nx)^n}{n!}S_n(x).$$

He claimed that

$$S_n(1)=\frac{n!}{2}\left(\frac{\mathrm{e}}{n}\right)^n-\frac{2}{3}+\frac{4}{135n}+O(n^{-2}),$$

which was independently proved in 1928 by Szegö and Watson. Buckholtz [6] found that

$$S_n(x)=\sum_{r=0}^{k-1}\frac{1}{n^r}U_r(x)+O(n^{-k}),$$

where

$$U_r(x)=(-1)^r\left(\frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^r\frac{x}{1-x}=(-1)^r\frac{C_r(x)}{(1-x)^{2r+1}},$$

and $C_r(x)$ is a polynomial of degree r. Let $\left\{{n \atop k}\right\}$ be the *Stirling number of the second kind*, i.e., the number of set partitions of $[n]$ into k blocks. In [7], Carlitz discovered that

$$\sum_{k=0}^{\infty}\left\{{n+k \atop k}\right\}x^k=\frac{C_n(x)}{(1-x)^{2n+1}}.$$

The polynomials $C_n(x)$ are now known as the *second-order Eulerian polynomials*.

Let $[\mathbf{n}]_2$ denote the multiset $\{1^2,2^2,\ldots,n^2\}$, where each element i appears 2 times. We say that the multipermutation σ of $[\mathbf{n}]_2$ is a *Stirling permutation* if $\sigma_i=\sigma_j$, then $\sigma_s>\sigma_i$ for all $i<s<j$. Let $\mathcal{Q}_n$ denote the set of all Stirling permutations of $[\mathbf{n}]_2$. For example, $\mathcal{Q}_2=\{1122,1221,2211\}$. Gessel-Stanley [17] discovered that $C_n(x)$ are the descent polynomials over all Stirling permutations in $\mathcal{Q}_n$. Recently, the theory of Stirling permutations has become an active research domain, see [5,21,24,25,32]. There are several variants of Stirling permutations, including Stirling permutations of a general multiset [23] and quasi-Stirling permutations [14].

For $\sigma\in\mathcal{Q}_n$, except where explicitly stated, we always assume that $\sigma_0=\sigma_{2n+1}=0$. Let

$$\begin{aligned}
\mathrm{Asc}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i\},\ \mathrm{Plat}(\sigma) = \{\sigma_i \mid \sigma_i = \sigma_{i+1}\},\ \mathrm{Des}(\sigma) = \{\sigma_i \mid \sigma_i > \sigma_{i+1}\},\\
\mathrm{Lap}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1}\},\ \mathrm{Rpd}(\sigma) = \{\sigma_i \mid \sigma_{i-1} = \sigma_i > \sigma_{i+1}\},\\
\mathrm{Eud}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_j > \sigma_{j+1},\ i < j\},\\
\mathrm{Apd}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} < \sigma_i = \sigma_{i+1} > \sigma_{i+2}\},\\
\mathrm{Vv}(\sigma) &= \{\sigma_i \mid \sigma_{i-1} > \sigma_i < \sigma_{i+1}, \sigma_{j-1} > \sigma_j < \sigma_{j+1},\ \sigma_i = \sigma_j,\ i < j-2\}
\end{aligned}$$

be the sets of ascents, plateaux, descents, left ascent-plateaux, right plateau-descents, exterior up-down-pairs, ascent-plateau-descents, valley-valley pairs of σ, respectively. We use $\mathrm{asc}\,(\sigma)$, $\mathrm{plat}\,(\sigma)$, $\mathrm{des}\,(\sigma)$, $\mathrm{lap}\,(\sigma)$, $\mathrm{rpd}(\sigma)$, $\mathrm{eud}(\sigma)$, $\mathrm{apd}(\sigma)$ and $\mathrm{vv}(\sigma)$ to denote the numbers of ascents, plateaux, descents, left ascent-plateaux, right plateau-descents, exterior up-down-pairs, ascent-plateau-descents, valley-valley pairs of σ, respectively. The statistic vv is a new statistic. It should be noted that if σ_i is a valley-valley pair value, then the two copies of σ_i are both valleys.

It is now well known that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{plat}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)}.$$

The *trivariate second-order Eulerian polynomials* are defined by

$$C_n(x,y,z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)} z^{\mathrm{des}\,(\sigma)}.$$

The study of $C_n(x,y,z)$ was initiated by Dumont [12], who discovered that

$$C_{n+1}(x,y,z) = xyz\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) C_n(x,y,z), \tag{3}$$

which implies that $C_n(x,y,z)$ is symmetric in its variables, i.e., it is unchanged under any permutation of the three variables. Bóna [2] independently found that the plateau statistic plat is equidistributed with the descent statistic des over $\mathcal{Q}_n$. The symmetry of the joint distribution (asc, des, plat) was rediscovered by Janson [20, Theorem 2.1]. In [19], Haglund-Visontai introduced a refinement of $C_n(x,y,z)$ by indexing each ascent, descent and plateau according to the values where they appear. Using the theory of context-free grammars, Chen-Fu [9] found the following result.

Proposition 2. *The trivariate polynomial $C_n(x,y,z)$ is e-positive, i.e.,*

$$C_n(x,y,z) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k}(x+y+z)^i(xy+yz+zx)^j(xyz)^k, \tag{4}$$

where the coefficient $\gamma_{n,i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.

Throughout this paper, let us assume that $\gamma_{n,i,j,k}$ is defined by (4). It follows from [9, eq. (4.9)] that

$$\gamma_{n,i,j,k} = 3(i+1)\gamma_{n-1,i+1,j,k-1} + 2(j+1)\gamma_{n-1,i-1,j+1,k-1} + k\gamma_{n-1,i,j-1,k},$$

with $\gamma_{1,0,0,1} = 1$ and $\gamma_{1,i,j,k} = 0$ if $k = 0$. For $n = 2, 3, 4$, the nonzero $\gamma_{n,i,j,k}$ are listed as follows:

$$\gamma_{2,0,1,1} = 1,\ \gamma_{3,1,0,2} = 2,\ \gamma_{3,0,2,1} = 1,\ \gamma_{4,0,0,3} = 6,\ \gamma_{4,1,1,2} = 8,\ \gamma_{4,0,3,1} = 1.$$

In [32], we introduced Stirling permutation codes and provided numerous equidistribution results as applications. The *trivariate ascent-plateau polynomials* are defined by

$$N_n(p,q,r) = \sum_{\sigma\in\mathcal{Q}_n} p^{\mathrm{lap}\,(\sigma)} q^{\mathrm{eud}(\sigma)} r^{\mathrm{rpd}(\sigma)}.$$

From [32, Theorem 21], we see that

$$N_n(p,q,r) = \sum_{i+2j+3k=2n+1} 3^i \gamma_{n,i,j,k}(p+q+r)^j (pqr)^k. \tag{5}$$

One may ask the following problem.

Problem 3. Why do (4) and (5) share the same coefficients?

Let

$$Q_n(x,y,z,p,q,r) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)} z^{\mathrm{des}\,(\sigma)} p^{\mathrm{lap}\,(\sigma)} q^{\mathrm{eud}(\sigma)} r^{\mathrm{rpd}(\sigma)}. \tag{6}$$

A special case of Theorem 13 gives an answer to Problem 3:

$$Q_n(x,y,z,p,q,r) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k}(x+y+z)^i (xyp+xzq+yzr)^j (xyzpqr)^k. \tag{7}$$

This paper is organized as follows. In Section 2, we first prove Theorem 1, and then we establish a strong connection between signed permutations and Stirling permutations. In Section 3, using SP-codes, we study a eight-variable polynomial $Q_n(x,y,z,p,q,r,s,t)$ as well as a seventeen-variable polynomial, where the parameter s marks the ascent-plateau-descent statistic, the parameter t marks the valley-valley pair statistic and

$$Q_n(x,y,z,p,q,r,1,1) = Q_n(x,y,z,p,q,r).$$

For the seventeen-variable polynomial, we find an expansion formula with the same coefficients as in (4), (5) and (7). Why the coefficients $\gamma_{n,i,j,k}$ play such a crucial role in these expansions is somewhat mysterious. From this paper, one can see that SP-code is the key to clarify it.

2. Eulerian polynomials and Stirling permutations

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. Following Chen [8], a *context-free grammar* over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces each letter in A by a formal function over A. The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u+v) = D_G(u) + D_G(v),\ D_G(uv) = D_G(u)v + uD_G(v).$$

In the theory of context-free grammars, there are two methods for studying combinatorics: grammatical labeling and the change of grammar, see [10,11,13,24,30,32] for various applications.

2.1. Proof of Theorem 1

Lemma 4. *If* $G = \{P \to PD + NA,\ N \to PD + NA,\ E \to (A+D)E,\ A \to 2AD,\ D \to 2AD\}$*, then for* $n \geqslant 1$*, we have*

$$D_G^{n-1}(PE+NE)|_{P=A=E=1,\ N=y,\ D=xy} = \sum_{\pi \in \mathfrak{S}_n^B} x^{\mathrm{des}_A(\pi)} y^{\mathrm{des}_B(\pi)}. \tag{8}$$

Proof. Given $\pi \in \mathfrak{S}_n^B$. We first give a grammatical labeling of π as follows:

- (i) We put a superscript P just before $\pi(1)$ if $\pi(1) > 0$, while we put a superscript N just before $\pi(1)$ if $\pi(1) < 0$;
- (ii) For $1 \leqslant i \leqslant n-1$, we put a superscript A right after $\pi(i)$ if $\pi(i) < \pi(i+1)$, while we put a superscript D right after $\pi(i)$ if $\pi(i) > \pi(i+1)$;
- (iii) Put a superscript E at the end of π.

With this labeling, the weight of π is defined as the product of its labels. Note that the sum of weights of elements in $\mathfrak{S}_1^B$ is given by $PE + NE$. In general, it is routine to verify that the insertion of n or $\overline{n}$ corresponds to one substitution rule given by G. For example, let $\pi = \overline{2}3\overline{1}54$. The labeling of π is given by ${}^N\overline{2}^A3^D\overline{1}^A5^D4^E$. If we insert 6 or $\overline{6}$ into π, we get

$${}^P6^D\overline{2}^A3^D\overline{1}^A5^D4^E,\ {}^N\overline{6}^A\overline{2}^A3^D\overline{1}^A5^D4^E,\ {}^N\overline{2}^A6^D3^D\overline{1}^A5^D4^E,\ {}^N\overline{2}^D\overline{6}^A3^D\overline{1}^A5^D4^E,$$
$${}^N\overline{2}^A3^A6^D\overline{1}^A5^D4^E,\ {}^N\overline{2}^A3^D\overline{6}^A\overline{1}^A5^D4^E,\ {}^N\overline{2}^A3^D\overline{1}^A6^D5^D4^E,\ {}^N\overline{2}^A3^D\overline{1}^D\overline{6}^A5^D4^E,$$
$${}^N\overline{2}^A3^D\overline{1}^A5^A6^D4^E,\ {}^N\overline{2}^A3^D\overline{1}^A5^D\overline{6}^A4^E,\ {}^N\overline{2}^A3^D\overline{1}^A5^D4^A6^E,\ {}^N\overline{2}^A3^D\overline{1}^A5^D4^D\overline{6}^E.$$

In conclusion, the action of the formal derivative D_G on the set of weighted signed permutations in $\mathfrak{S}_n^B$ gives the set of weighted signed permutations in $\mathfrak{S}_{n+1}^B$. Substituting $P = A = E = 1,\ N = y$ and $D = xy$, we get the desired result. □

A proof of Theorem 1:

Proof. Let G be the grammar given by Lemma 4. Consider a change of G. Setting $P + N = a,\ b = PD + NA,\ c = AD$ and $d = A + D$, we see that

$$D_G(a) = 2b,\ D_G(b) = 2ac + bd,\ D_G(c) = 2cd,\ D_G(d) = 4c,\ D_G(E) = dE.$$

Thus we get a new grammar: $G' = \{a \to 2b,\ b \to 2ac + bd,\ c \to 2cd,\ d \to 4c,\ E \to dE\}$. Note that $D_{G'}(aE) = adE + 2bE$ and $D^2_{G'}(aE) = aE(d^2 + 8c) + bE(6d)$. For $n \geqslant 2$, by induction, we find that there exist nonnegative integers $\xi(n,k)$ and $\zeta(n,k)$ such that

$$D^{n-1}_{G'}(aE) = aE \sum_{k\geqslant 0} 4^k \xi(n,k) c^k d^{n-1-2k} + bE \sum_{k\geqslant 0} 4^k \zeta(n,k) c^k d^{n-2-2k}. \tag{9}$$

Using $D^n_{G'}(aE) = D_{G'}\left(D^{n-1}_{G'}(aE)\right)$, it is easy to verify that

$$\begin{cases} \xi(n+1,k) = (1+2k)\xi(n,k) + (n-2k+1)\xi(n,k-1) + \frac{1}{2}\zeta(n,k-1), \\ \zeta(n+1,k) = 2(1+k)\zeta(n,k) + (n-2k)\zeta(n,k-1) + 2\xi(n,k), \end{cases} \tag{10}$$

with $\xi(1,0) = 1$ and $\xi(1,k) = 0$ if $k \neq 0$ and $\zeta(1,k) = 0$ for any k. In (9), upon taking $a = 1 + y,\ b = xy + y,\ E = 1,\ c = xy$ and $d = 1 + xy$, we get the desired expansion formula.

Let

$$T_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{udrun}\,(\pi)} = \sum_{k=1}^{n} T(n,k)x^k.$$

The polynomials $T_n(x)$ satisfy the recursion

$$T_{n+1}(x) = x(nx+1)T_n(x) + x\left(1 - x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}T_n(x), \tag{11}$$

with $T_0(x) = 1$ and $T_1(x) = x$ (see [31,38]). If we set $T_n(x) = \frac{x}{2}\widehat{T}_n(x)$, then it follows from (11) that $\widehat{T}_{n+1}(x) = (1 + x + (n-1)x^2)\widehat{T}_n(x) + x\left(1 - x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}\widehat{T}_n(x)$.

Consider the following three polynomials

$$\xi_n(x) = \sum_{k\geqslant 0} \xi(n,k)x^k,\ \zeta_n(x) = \sum_{k\geqslant 0} \zeta(n,k)x^k,\ f_n(x) = 2\xi_n(x^2) + x\zeta_n(x^2). \tag{12}$$

From (13), we see that $f_n(x)$ satisfy the same recurrence relation and initial conditions as $\widehat{T}_n(x)$, so they agree. Therefore, we deduce that

$$T_n(x) = \frac{x}{2} f_n(x) = \frac{x}{2} \left(2\xi_n(x^2) + x\zeta_n(x^2) \right),$$

which yields $\xi(n,k) = T(n, 2k+1)$ and $\zeta(n,k) = 2T(n, 2k+2)$. This completes the proof. □

2.2. Real-rooted polynomials

Let $\xi_n(x)$ and $\zeta_n(x)$ be defined by (12). Multiplying both sides of (10) by x^i and summing over all i, we arrive at the following result.

Proposition 5. *Let $\xi_1(x) = 1$ and $\zeta_1(x) = 0$. Then we have*

$$\begin{cases} \xi_{n+1}(x) = (1 + (n-1)x)\xi_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\xi_n(x) + \frac{x}{2}\zeta_n(x), \\ \zeta_{n+1}(x) = (2 + (n-2)x)\,\zeta_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\zeta_n(x) + 2\xi_n(x). \end{cases}$$

In particular, $\xi_2(x) = 1$, $\zeta_2(x) = 2$, $\xi_3(x) = 1 + 2x$, $\zeta_3(x) = 6$. Moreover, we have $\xi_n(1) = \frac{n!}{2}$ and $\zeta_n(1) = n!$. Let RZ denote the set of real polynomials with only real zeros. Furthermore, denote by $\mathrm{RZ}(I)$ the set of such polynomials all of whose zeros are in the interval I.

Theorem 6. *Let $f_n(x) = 2\xi_n(x^2) + x\zeta_n(x^2)$. The polynomials $f_n(x)$ satisfy the recursion*

$$f_{n+1}(x) = (1 + x + (n-1)x^2) f_n(x) + x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} f_n(x), \; f_1(x) = 2. \tag{13}$$

Then $f_n(x) \in \mathrm{RZ}[-1, 0)$ and $f_n(x)$ interlaces $f_{n+1}(x)$. More precisely, $f_n(x)$ has $\lfloor \frac{n-1}{2} \rfloor$ simple zeros in the interval $(-1, 0)$ and the zero $x = -1$ with the multiplicity $\lfloor \frac{n}{2} \rfloor$. Moreover, both $\xi_n(x)$ and $\zeta_n(x)$ have only real negative simple zeros, $\zeta_{2n}(x)$ alternates left of $\xi_{2n}(x)$ and $\zeta_{2n+1}(x)$ interlaces $\xi_{2n+1}(x)$.

Below are the polynomials $f_n(x)$ for $n = 2, 3, 4$:

$$f_2(x) = 2 + 2x, \; f_3(x) = 2 + 6x + 4x^2, \; f_4(x) = 2 + 14x + 22x^2 + 10x^3.$$

In the sequel, we shall prove Theorem 6. Following [16], we say that a polynomial $p(x) \in \mathbb{R}[x]$ is *standard* if its leading coefficient is positive. Suppose that $p(x), q(x) \in \mathrm{RZ}$. The zeros of $p(x)$ are $\xi_1 \leqslant \cdots \leqslant \xi_n$, and that those of $q(x)$ are $\theta_1 \leqslant \cdots \leqslant \theta_m$. We say that $p(x)$ *interlaces* $q(x)$ if $\deg q(x) = 1 + \deg p(x)$ and the zeros of $p(x)$ and $q(x)$ satisfy $\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_{n+1}$. We say that $p(x)$ *alternates left of* $q(x)$ if $\deg p(x) = \deg q(x)$ and the zeros of them satisfy $\xi_1 \leqslant \theta_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_n$. We use the notation $p(x) \prec q(x)$ for either $p(x)$ interlaces $q(x)$ or $p(x)$ alternates left of $q(x)$. A complex coefficient polynomial $p(x)$ is said to be *weakly Hurwitz stable* if every zero of $p(x)$ is in the closed left half of the complex plane. The following version of the Hermite-Biehler theorem will be used in the proof of Theorem 6.

Hermite-Biehler Theorem *([16, Theorem 3]). Let $f(x)=f^E(x^2)+xf^O(x^2)$ be a standard polynomial with real coefficients. Then $f(x)$ is weakly Hurwitz stable if and only if both $f^E(x)$ and $f^O(x)$ are standard, have only nonpositive zeros, and $f^O(x)\prec f^E(x)$.*

A proof of Theorem 6:

Proof. Setting $a_n(x)=2\xi_n(x)$, it follows from Proposition 5 that

$$\begin{cases} a_{n+1}(x)=(1+(n-1)x)a_n(x)+2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}a_n(x)+x\zeta_n(x), \\ \zeta_{n+1}(x)=(2+(n-2)x)\,\zeta_n(x)+2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\zeta_n(x)+a_n(x). \end{cases}$$

Since $f_n(x)=2\xi_n(x^2)+x\zeta_n(x^2)=a_n(x^2)+x\zeta_n(x^2)$, then the recursion (13) follows from [31, Theorem 3]. When $n\geqslant 2$, it is clear that $f_n(0)=a_n(0)=a_1(0)=2$, $\deg(a_n(x))=\lfloor\frac{n-1}{2}\rfloor$ and $\deg(\zeta_n(x))=\lfloor\frac{n-2}{2}\rfloor$. By [27, Theorem 2], we see that $f_n(x)$ interlaces $f_{n+1}(x)$, $f_n(x)$ has $\lfloor\frac{n-1}{2}\rfloor$ simple zeros in the interval $(-1,0)$ and the zero $x=-1$ with the multiplicity $\lfloor\frac{n}{2}\rfloor$. It follows from Hermite-Biehler Theorem that both $a_n(x)$ and $\zeta_n(x)$ have only real negative simple zeros, $\zeta_{2n}(x)$ alternates left of $a_{2n}(x)$ and $\zeta_{2n+1}(x)$ interlaces $a_{2n+1}(x)$. This completes the proof. □

2.3. Relationship between signed permutations and Stirling permutations

Let $\mathcal{Q}_n^{(1)}=\{\sigma\in\mathcal{Q}_n\mid \text{the two copies of 1 are contiguous elements in } \sigma\}$. For example,

$$\mathcal{Q}_1^{(1)}=\{11\},\quad \mathcal{Q}_2^{(1)}=\{1122,\ 2211\},$$
$$\mathcal{Q}_3^{(1)}=\{112233,\ 112332,\ 113322,\ 331122,\ 221133,\ 223311,\ 233211,\ 332211\}.$$

Given $\sigma\in\mathcal{Q}_n^{(1)}$. Let us examine how to generate an element in $\mathcal{Q}_{n+1}^{(1)}$ by inserting the two copies of $n+1$. Note that there are $2n$ possibilities. Thus $\#\mathcal{Q}_{n+1}^{(1)}=2n\#\mathcal{Q}_n^{(1)}=2^nn!$. Recall that $\#\mathfrak{S}_n^B=2^nn!$. It is natural to explore the relationship between $\mathcal{Q}_{n+1}^{(1)}$ and $\mathfrak{S}_n^B$. We need some notations. For $\sigma\in\mathcal{Q}_n$, a value σ_i is called

- an *ascent-plateau* if $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, where $2\leqslant i\leqslant 2n-1$;
- a *left ascent-plateau* if $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, where $1\leqslant i\leqslant 2n-1$ and $\sigma_0=0$.

Let $\mathrm{ap}\,(\sigma)$ (resp. $\mathrm{lap}\,(\sigma)$) be the number of ascent-plateaux (resp. left ascent-plateaux) in σ. The *ascent-plateau polynomials* (also called 1/2-*Eulerian polynomials*) $M_n(x)$ [29,36] and the *left ascent-plateau polynomials* $\widetilde{M}_n(x)$ [25,29] can be defined as follows:

$$M_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{ap}\,(\sigma)},\ \widetilde{M}_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{lap}\,(\sigma)}.$$

From [28, Proposition 1], we see that

$$2^n x A_n(x)=\sum_{i=0}^{n}\binom{n}{i}\widetilde{M}_i(x)\widetilde{M}_{n-i}(x),\ B_n(x)=\sum_{i=0}^{n}\binom{n}{i}M_i(x)\widetilde{M}_{n-i}(x).$$

Let $\mathcal{Q}_{n+1}^{(0)}$ be the set of Stirling permutations of the multiset $\{1,2^2,3^2,\ldots,n^2,(n+1)^2\}$, where only the element 1 appears one time. Note that $\mathcal{Q}_n^{(0)}\cong\mathcal{Q}_n^{(1)}$. So $\#\mathcal{Q}_{n+1}^{(0)}=\#\mathcal{Q}_{n+1}^{(1)}=2^n n!$. For $\sigma\in\mathcal{Q}_{n+1}^{(0)}$, we say that σ_i is an *even value* if the first appearance of σ_i occurs at an even position of σ, where $\sigma_i\in\{2,3,\ldots,n+1\}$. Let $\mathrm{even}\,(\sigma)$ be the number of even indices in σ. For example, $\mathrm{even}\,(1\mathbf{2}2)=1,\ \mathrm{even}\,(221)=0$ and $\mathrm{even}\,(2\mathbf{3}321)=1$.

Theorem 7. *For $n\geqslant 1$, we have*

$$\sum_{\pi\in\mathfrak{S}_n^B}x^{\mathrm{des}_A(\pi)+1}y^{\mathrm{des}_B(\pi)}q^{\mathrm{neg}\,(\pi)}=\sum_{\sigma\in\mathcal{Q}_{n+1}^{(0)}}x^{\mathrm{lap}\,(\sigma)}y^{\mathrm{ap}\,(\sigma)}q^{\mathrm{even}\,(\sigma)}. \tag{14}$$

Proof. Using the same labeling scheme for $\pi\in\mathfrak{S}_n^B$ as in the proof of Lemma 4, and attaching a weight q to each negative element, one can easily verify that

$$D_{G_1}^{n-1}(PE+qNE)|_{P=x,\ A=E=1,\ N=xy,\ D=xy}=\sum_{\pi\in\mathfrak{S}_n^B}x^{\mathrm{des}_A(\pi)+1}y^{\mathrm{des}_B(\pi)}q^{\mathrm{neg}\,(\pi)}, \tag{15}$$

where the grammar G_1 is defined by $G_1=\{P\to PD+qNA,\ N\to PD+qNA,\ E\to(A+qD)E,\ A\to(1+q)AD,\ D\to(1+q)AD\}$.

Given $\sigma\in\mathcal{Q}_n^{(0)}$. We now introduce a labeling scheme for σ:

- (i) If $\sigma_1=\sigma_2$, then σ_1 is a left ascent-plateau, and we label the two positions just before and right after σ_1 by a subscript label α. For any other ascent-plateau σ_i, i.e., $i\geqslant 2$ and $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, we label the two positions just before and right after σ_i by a label β;
- (ii) If $\sigma_1<\sigma_2$, then we use the superscript γ to mark the first position (just before σ_1) and the last position (at the end of σ), and denoted by $\overbrace{\sigma_1\sigma_2\cdots\sigma_{2n}}^{\gamma}$;
- (iii) For any other element, we label it by a subscript w;
- (iv) We attach a superscript label q to every even value.

As illustrations, the labeled elements in $\mathcal{Q}_2^{(0)}$ can be listed as follows:

$$\overbrace{1\underbrace{2^q}_{\beta}2}^{\gamma},\ \underbrace{2}_{\alpha}2_w1_w.$$

Let us examine how to generate the elements in $\mathcal{Q}_3^{(0)}$ by inserting two copies of 3:

$$\overbrace{1\underbrace{2^q}_{\beta}2}^{\gamma}\begin{cases}\underbrace{3}_{\alpha}3_w1\underbrace{2^q}_{\beta}2_w,\\ \overbrace{1\underbrace{3^q}_{\beta}3_w2^q_w2}^{\gamma},\\ \overbrace{1_w2^q\underbrace{3}_{\beta}3_w2}^{\gamma},\\ \overbrace{1\underbrace{2^q}_{\beta}2\underbrace{3^q}_{\beta}3}^{\gamma},\end{cases}\qquad \underbrace{2}_{\alpha}2_w1_w\to\begin{cases}\underbrace{3}_{\alpha}3_w2_w2_w1_w,\\ \overbrace{2\underbrace{3^q}_{\beta}3_w2_w1}^{\gamma},\\ \underbrace{2}_{\alpha}2\underbrace{3}_{\beta}3_w1_w,\\ \underbrace{2}_{\alpha}2_w1\underbrace{3^q}_{\beta}3_w.\end{cases} \tag{16}$$

Note that the labels w always appear even times. So we can read the labels w two by two from left to right, and set $W = w^2$. In the other words, we use W to record pairwise nearest elements with labels w from left to right. Note that the sum of weights of elements in $\mathcal{Q}_2^{(0)}$ is given by $\alpha w^2 + q\beta\gamma = \alpha W + q\beta\gamma$. By induction, as illustrated by (16), we see that if

$$G_2 = \{\alpha \to \alpha W + q\beta\gamma,\ \gamma \to \alpha W + q\beta\gamma,\ \beta \to (1+q)\beta W,\ W \to (1+q)\beta W\},$$

then

$$D_{G_2}^{n-1}(\alpha W + q\beta\gamma)|_{\alpha=x,\ \gamma=W=1,\ \beta=xy} = \sum_{\sigma\in\mathcal{Q}_{n+1}^{(0)}} x^{\mathrm{lap}\,(\sigma)}y^{\mathrm{ap}\,(\sigma)}q^{\mathrm{even}\,(\sigma)}. \tag{17}$$

Note that

$$D_{G_1}(PE + qNE) = PE(A + D + 2qD) + qNE(2A + qA + qD),$$
$$D_{G_2}(\alpha W + q\beta\gamma) = \alpha W(W + \beta + 2q\beta) + q\beta\gamma(2W + qW + q\beta).$$

When $n \geqslant 1$, by induction, it is routine to check that

$$D_{G_1}^n(PE + qNE) = PEf_n(A, D; q) + qNEg_n(A, D; q),$$
$$D_{G_2}^n(\alpha W + q\beta\gamma) = \alpha W f_n(W, \beta; q) + q\beta\gamma g_n(W, \beta; q),$$

where $f_n(x, y; q)$ and $g_n(x, y; q)$ satisfy the recurrence system:

$$\begin{cases} f_{n+1}(x, y; q) = (x + y + qy)f_n(x, y; q) + D_{G_3}(f_n(x, y; q)) + qyg_n(x, y; q),\\ g_{n+1}(x, y; q) = (x + qx + qy)g_n(x, y; q) + D_{G_3}(g_n(x, y; q)) + xf_n(x, y; q),\end{cases}$$

with $f_1(x, y; q) = g_1(x, y; q) = 1$ and $G_3 = \{x \to (1+q)xy,\ y \to (1+q)xy\}$. Therefore, if we make the substitutions: $PE \Leftrightarrow \alpha W,\ NE \Leftrightarrow \beta\gamma,\ A \Leftrightarrow W,\ D \Leftrightarrow \beta$, we obtain

$$D_{G_1}^{n-1}(PE + qNE)|_{P=x,\ A=E=1,\ N=xy,\ D=xy} = D_{G_2}^{n-1}(\alpha\gamma + q\beta\delta)|_{\alpha=x,\ \gamma=W=1,\ \beta=xy},$$

and so we arrive at (14). This completes the proof. □

For $\pi \in \mathfrak{S}_n$, an *excedance* of π is an index i such that $\pi(i) > i$. Let $\operatorname{exc}(\pi)$ be the number of excedances of π. A permutation is called a *derangement* if it has no fixed points. Let $\mathcal{D}_n$ be the set of all derangements in $\mathfrak{S}_n$. The *derangement polynomials* and the *binomial-Eulerian polynomials* are respectively defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\operatorname{exc}(\pi)}, \; \widetilde{A}_n(x) = 1 + x \sum_{k=1}^{n} \binom{n}{k} A_k(x).$$

The first few of $d_n(x)$ and $\widetilde{A}_n(x)$ are given as follows:

$$d_1(x) = 0, \; d_2(x) = x, \; d_3(x) = x + x^2, \; d_4(x) = x + 7x^2 + x^3;$$
$$\widetilde{A}_1(x) = 1 + x, \; \widetilde{A}_2(x) = 1 + 3x + x^2, \; \widetilde{A}_3(x) = 1 + 7x + 7x^2 + x^3.$$

The reader is referred to [22] for the recent progress on the binomial-Eulerian polynomials.

Consider the following polynomials

$$b_n(x, y, q) = \sum_{\sigma \in \mathcal{Q}_{n+1}^{(0)}} x^{\operatorname{lap}(\sigma)} y^{\operatorname{ap}(\sigma)} q^{\operatorname{even}(\sigma)}.$$

Combining (14), [4, Eq. (14)] and [26, Eq. (5)], we discover the following result.

Corollary 8. *We have*

$$b_n(1, y, -1) = \sum_{\sigma \in \mathcal{Q}_{n+1}^{(0)}} y^{\operatorname{ap}(\sigma)} (-1)^{\operatorname{even}(\sigma)} = (1 - y)^n,$$
$$b_n\left(1, y, -\frac{1}{y}\right) = \sum_{\sigma \in \mathcal{Q}_{n+1}^{(0)}} y^{\operatorname{ap}(\sigma) - \operatorname{even}(\sigma)} (-1)^{\operatorname{even}(\sigma)} = y \left(\frac{y-1}{y}\right)^n d_n(y),$$
$$b_n\left(1, y, -y\right) = \sum_{\sigma \in \mathcal{Q}_{n+1}^{(0)}} y^{\operatorname{ap}(\sigma) + \operatorname{even}(\sigma)} (-1)^{\operatorname{even}(\sigma)} = (1 - y)^n \widetilde{A}_n(y).$$

The *type D Coxeter group* $\mathfrak{S}_n^D$ is the subgroup of $\mathfrak{S}_n^B$ consisting of signed permutations with an even number of negative entries. It follows from Theorem 7 that

$$\#\{\sigma \in \mathcal{Q}_{n+1}^{(0)} \mid \operatorname{even}(\sigma) \text{ is odd}\} = \#\{\sigma \in \mathcal{Q}_{n+1}^{(0)} \mid \operatorname{even}(\sigma) \text{ is even}\} = \#\mathfrak{S}_n^D = 2^{n-1} n!.$$

Let $\operatorname{des}_D(\pi) = \#\{i \in [n] \mid \pi(i-1) > \pi(i)\}$, where $\pi(0) = -\pi(2)$. The *type D Eulerian polynomial* is defined by

$$D_n(x) = \sum_{\pi \in \mathfrak{S}_n^D} x^{\mathrm{des}_D(\pi)}.$$

Stembridge [39, Lemma 9.1] obtained that $D_n(x) = B_n(x) - n2^{n-1}xA_{n-1}(x)$ for $n \geqslant 2$. We end this section by posing two problems.

Problem 9. Could we find a combinatorial interpretation of the type D Eulerian polynomial $D_n(x)$ over the set $\{\sigma \in \mathcal{Q}_{n+1}^{(0)} \mid \mathrm{even}\,(\sigma) \text{ is even}\}$?

Problem 10. How to relate $\mathcal{Q}_n^{(0)}$ and $\{\sigma \in \mathcal{Q}_n^{(0)} \mid \mathrm{even}\,(\sigma) \text{ is even}\}$ to group operations?

3. Eight-variable and seventeen-variable polynomials

3.1. Definitions, notation and preliminary results

In equivalent forms, Dumont [12], Haglund-Visontai [19], Chen-Hao-Yang [10] and Ma-Ma-Yeh [30] all showed that

$$D_G^n(x) = C_n(x, y, z),$$

where $G = \{x \to xyz,\ y \to xyz,\ z \to xyz\}$. By the change of grammar $u = x+y+z$, $v = xy + yz + zx$ and $w = xyz$, it is clear that $D_G(u) = 3w$, $D_G(v) = 2uw$, $D_G(w) = vw$. So we get a grammar

$$H = \{w \to vw, u \to 3w, v \to 2uw\}. \tag{18}$$

For any $n \geqslant 1$, Chen-Fu [9] discovered that

$$C_n(x, y, z) = D_G^n(x) = D_H^{n-1}(w) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} u^i v^j w^k. \tag{19}$$

Substituting $u \to x + y + z$, $v \to xy + yz + zx$ and $w \to xyz$, one can immediately obtain (4).

A *rooted tree* of order n with the vertices labeled $1, 2, \ldots, n$, is an increasing tree if the node labeled 1 is distinguished as the root, and the labels along any path from the root are increasing. An *increasing plane tree* is an increasing tree with the children of each vertex are linearly ordered.

Definition 11. A *ternary increasing tree* of size n is an increasing plane tree with $3n + 1$ nodes in which each interior node has a label and three children (a left child, a middle child and a right child), and exterior nodes have no children and no labels.

Let $\mathcal{T}_n$ denote the set of ternary increasing trees of size n, see Fig. 1 for instance. For any $T \in \mathcal{T}_n$, it is clear that T has exactly $2n + 1$ exterior nodes. Let $\mathrm{exl}(T)$

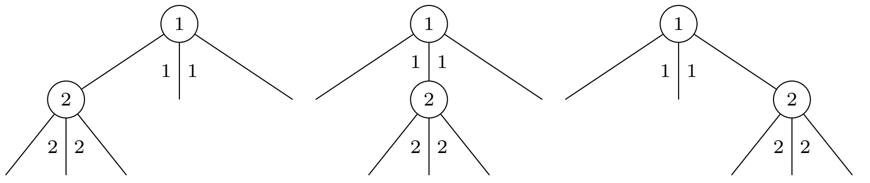

Fig. 1. The ternary increasing trees of order 2 encoded by 2211, 1221, 1122, and their SP-codes are given by $((0,0),(1,1))$, $((0,0)(1,2))$ and $((0,0)(1,3))$, respectively.

(resp. $\operatorname{exm}(T)$, $\operatorname{exr}(T)$) be the number of exterior left nodes (resp. exterior middle nodes, exterior right nodes) in T. Using a recurrence relation that is equivalent to (3), Dumont [12, Proposition 1] found that

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)}y^{\operatorname{plat}(\sigma)}z^{\operatorname{des}(\sigma)}=\sum_{T\in\mathcal{T}_n} x^{\operatorname{exl}(T)}y^{\operatorname{exm}(T)}z^{\operatorname{exr}(T)}. \tag{20}$$

A bijective proof of (20) can be found in the proof of [21, Theorem 1]. For the convenience of the reader, we present a brief description of it. Given $T\in\mathcal{T}_n$. Between the 3 edges of T going out from a node labeled v, we place 2 integers v. Now we perform the depth-first traversal and encode the tree T by recording the sequence of the labels visited as we traverse around T. The encoded sequence, denoted as $\phi(T)$, is a Stirling permutation. Given $\sigma\in\mathcal{Q}_n$. We proceed recursively by decomposing σ as $u_1 1 u_2 1 u_3$, where the u_i's are again Stirling permutations. The smallest label in each u_i is attached to the root node labeled 1. One can recursively apply this procedure to each u_i to obtain the tree representation, and $\phi^{-1}(\sigma)$ is a ternary increasing tree. Using ϕ, one can immediately find that the identity (20) holds.

For $\sigma\in\mathcal{Q}_n$, let

$$\begin{aligned}
\operatorname{Dplat}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}>\sigma_i=\sigma_{i+1}\},\ \operatorname{Dasc}(\sigma)=\{\sigma_i\mid\sigma_{i-1}<\sigma_i<\sigma_{i+1}\},\\
\operatorname{Dd}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}>\sigma_i=\sigma_j>\sigma_{j+1},\ i<j\},\\
\operatorname{Uu}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}<\sigma_i=\sigma_j<\sigma_{j+1},\ i<j\},\\
\operatorname{Ddes}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}>\sigma_i>\sigma_{i+1}\},\ \operatorname{Pasc}(\sigma)=\{\sigma_i\mid\sigma_{i-1}=\sigma_i<\sigma_{i+1}\},\\
\operatorname{Dav}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}<\sigma_i<\sigma_{i+1}\ \&\ \sigma_{j-1}>\sigma_j<\sigma_{j+1}\ \&\ \sigma_i=\sigma_j,\ i<j\},\\
\operatorname{Dpa}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}>\sigma_i=\sigma_{i+1}<\sigma_{i+2}\},\\
\operatorname{Vdd}(\sigma)&=\{\sigma_i\mid\sigma_{i-1}>\sigma_i<\sigma_{i+1}\ \&\ \sigma_{j-1}>\sigma_j>\sigma_{j+1}\ \&\ \sigma_i=\sigma_j,\ i<j\}.
\end{aligned}$$

denote the sets of descent-plateaux, double ascents, down-down pairs, up-up pairs, double descents, plateau-ascents, double ascent-valley pairs, descent-plateau-ascents and valley-double descent pairs of σ, respectively. Let $\operatorname{dplat}(\sigma)$ (resp. $\operatorname{dasc}(\sigma)$, $\operatorname{dd}(\sigma)$, $\operatorname{uu}(\sigma)$, $\operatorname{ddes}(\sigma)$, $\operatorname{pasc}(\sigma)$, $\operatorname{dav}(\sigma)$ $\operatorname{dpa}(\sigma)$, $\operatorname{vdd}(\sigma)$) denote the number of descent-plateaux (resp. double ascents, down-down pairs, up-up pairs, double descents, plateau-ascents, double ascent-valley pairs, descent-plateau-ascents, valley-double descent pairs) in σ. It should be noted that the statistics dav, dpa, vdd are all new statistics.

In the sequel, we give a summary of Stirling permutation codes. Following [32, p. 10], any ternary increasing tree of size n can be built up from the root 1 by successively adding nodes $2, 3, \ldots, n$. Clearly, node 2 is a child of the root 1 and the root 1 has at most three children. For $2 \leqslant i \leqslant n$, when node i is inserted, we distinguish three cases:

(c_1) if it is the left child of a node $v \in [i-1]$, then the node i is coded as $[v, 1]$;
(c_2) if it is the middle child of a node $v \in [i-1]$, then the node i is coded as $[v, 2]$;
(c_3) if it is the right child of a node $v \in [i-1]$, then the node i is coded as $[v, 3]$.

Thus the node i is coded as a 2-tuple (a_{i-1}, b_{i-1}), where $1 \leqslant a_{i-1} \leqslant i-1$, $1 \leqslant b_{i-1} \leqslant 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leqslant i < j \leqslant n-1$. For convenience, we name this build-tree code as Stirling permutation code. By convention, the root 1 is coded as $(0, 0)$.

Definition 12 *([32]).* A 2-tuples sequence $C_n = ((0,0), (a_1, b_1), (a_2, b_2) \ldots, (a_{n-1}, b_{n-1}))$ of length n is called a Stirling permutation code (SP-code for short) if $1 \leqslant a_i \leqslant i$, $1 \leqslant b_i \leqslant 3$ and $(a_i, b_i) \neq (a_j, b_j)$ for all $1 \leqslant i < j \leqslant n-1$.

Let CQ_n be the set of SP-codes of length n. In particular, we have

$$\mathrm{CQ}_1 = \{(0,0)\},\ \mathrm{CQ}_2 = \{(0,0)(1,1),\ (0,0)(1,2),\ (0,0)(1,3)\}.$$

We now describe a bijection Γ between $\mathcal{Q}_n$ and CQ_n. There are three cases to obtain an element of $\mathcal{Q}_n$ from an element $\sigma \in \mathcal{Q}_{n-1}$ by inserting the two copies of n between σ_i and σ_{i+1}: $\sigma_i < \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$ or $\sigma_i > \sigma_{i+1}$. Set $\Gamma(11) = (0,0)$. When $n \geqslant 2$, the bijection $\Gamma : \mathcal{Q}_n \to \mathrm{CQ}_n$ can be defined as follows:

(c_1) $\sigma_i < \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 1)$;
(c_2) $\sigma_i = \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_{i+1}, 2)$;
(c_3) $\sigma_i > \sigma_{i+1}$ if and only if $(a_{n-1}, b_{n-1}) = (\sigma_i, 3)$.

As discussed in [32], combining the bijections ϕ (from Stirling permutations to ternary increasing trees) and Γ (from ternary increasing trees to Stirling permutation codes), we obtain Table 1, which contains the correspondences of set valued statistics on Stirling permutations and SP-codes. It should be noted that the last four are new correspondences.

3.2. Main results

In order to study the six-variable polynomials defined by (6), we find that it would be necessary to introduce the following eight-variable polynomials:

Table 1
The correspondences of statistics on Stirling permutations and SP-codes.

Statistics on Stirling permutation	Statistics on SP-code
Asc (ascent)	$[n]-\{a_i \mid (a_i,1)\in C_n\}$
Plat (plateau)	$[n]-\{a_i \mid (a_i,2)\in C_n\}$
Des (descent)	$[n]-\{a_i \mid (a_i,3)\in C_n\}$
Lap (left ascent-plateau)	$[n]-\{a_i \mid (a_i,1)$ or $(a_i,2)\in C_n\}$
Rpd (right plateau-descent)	$[n]-\{a_i \mid (a_i,2)$ or $(a_i,3)\in C_n\}$
Eud (exterior up-down pair)	$[n]-\{a_i \mid (a_i,1)$ or $(a_i,3)\in C_n\}$
Dasc (double ascent)	$\{a_i \mid (a_i,1)\notin C_n$ & $(a_i,2)\in C_n\}$
Dplat (descent-plateau)	$\{a_i \mid (a_i,1)\in C_n$ & $(a_i,2)\notin C_n\}$
Ddes (double descent)	$\{a_i \mid (a_i,2)\in C_n$ & $(a_i,3)\notin C_n\}$
Pasc (plateau-ascent)	$\{a_i \mid (a_i,2)\notin C_n$ & $(a_i,3)\in C_n\}$
Uu (up-up pair)	$\{a_i \mid (a_i,1)\notin C_n$ & $(a_i,3)\in C_n\}$
Dd (down-down pair)	$\{a_i \mid (a_i,1)\in C_n$ & $(a_i,3)\notin C_n\}$
Apd (ascent-plateau-descent)	$\{a_i \mid (a_i,1)\notin C_n$ & $(a_i,2)\notin C_n$ & $(a_i,3)\notin C_n\}$
Vv (valley-valley pair)	$\{a_i \mid (a_i,1)\in C_n$ & $(a_i,2)\in C_n$ & $(a_i,3)\in C_n\}$
Dav (double ascent-valley pair)	$\{a_i \mid (a_i,1)\notin C_n$ & $(a_i,2)\in C_n$ & $(a_i,3)\in C_n\}$
Dpa (descent-plateau-ascent)	$\{a_i \mid (a_i,1)\in C_n$ & $(a_i,2)\notin C_n$ & $(a_i,3)\in C_n\}$
Vdd (valley-double descent pair)	$\{a_i \mid (a_i,1)\in C_n$ & $(a_i,2)\in C_n$ & $(a_i,3)\notin C_n\}$

$$Q_n(x,y,z,p,q,r,s,t)=\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)}y^{\operatorname{plat}(\sigma)}z^{\operatorname{des}(\sigma)}p^{\operatorname{lap}(\sigma)}q^{\operatorname{eud}(\sigma)}r^{\operatorname{rpd}(\sigma)}s^{\operatorname{apd}(\sigma)}t^{\operatorname{vv}(\sigma)}.$$

where $\operatorname{apd}(\sigma)$ and $\operatorname{vv}(\sigma)$ are the numbers of ascent-plateau-descents and valley-valley pairs of σ, respectively. In particular, $Q_1=xyzpqrs$ and $Q_2=xyzpqrs(xyp+xzq+yzr)$. For convenience, we set $Q_n:=Q_n(x,y,z,p,q,r,s,t)$. We are now ready to answer Problem 3.

Theorem 13. *For any $n\geqslant 1$, we have the following decomposition*

$$\begin{aligned}&Q_n(x,y,z,p,q,r,s,t)\\&\quad=t^n\sum_{i+2j+3k=2n+1}\gamma_{n,i,j,k}\left(\frac{x+y+z}{t}\right)^i\left(\frac{xyp+xzq+yzr}{t}\right)^j\left(\frac{xyzpqrs}{t}\right)^k.\end{aligned}$$

We next relate the polynomial Q_n to a five-variable polynomial.

Theorem 14. *The polynomials $Q_n:=Q_n(x,y,z,p,q,r,s,t)$ can be expanded as follows:*

$$\begin{aligned}Q_n=&\left(\frac{x+y+z}{3}\right)^n\\&\times Q_n\left(1,1,1,\frac{3xyp}{x+y+z},\frac{3xzq}{x+y+z},\frac{3yzr}{x+y+z},\frac{s(x+y+z)^2}{9xyz},\frac{3t}{x+y+z}\right).\end{aligned}$$

Since $C_n(x,y,z)=Q_n(x,y,z,1,1,1,1,1)$, we obtain

$$C_n(x,y,z)=\sum_{\sigma\in\mathcal{Q}_n}(xy)^{\operatorname{lap}(\sigma)}(xz)^{\operatorname{eud}(\sigma)}(yz)^{\operatorname{rpd}(\sigma)}(xyz)^{-\operatorname{apd}(\sigma)}\left(\frac{x+y+z}{3}\right)^{\alpha_n(\sigma)},\quad (21)$$

where $\alpha_n(\sigma) = n + 2\,\mathrm{apd}(\sigma) - \mathrm{lap}\,(\sigma) - \mathrm{eud}(\sigma) - \mathrm{rpd}(\sigma) - \mathrm{vv}(\sigma)$.

It should be noted that Theorem 13 implies that the following two results are equivalent:

- the triple statistic $(\mathrm{asc}\,, \mathrm{plat}\,, \mathrm{des}\,)$ is a symmetric distribution over $\mathcal{Q}_n$;
- the triple statistic $(\mathrm{lap}\,, \mathrm{eud}, \mathrm{rpd})$ is a symmetric distribution over $\mathcal{Q}_n$.

Define

$$M_n(\beta_1, \beta_3, \beta_5) = \sum_{\sigma \in \mathcal{Q}_n} \beta_1^{\mathrm{dplat}(\sigma)} \beta_4^{\mathrm{uu}(\sigma)} \beta_5^{\mathrm{ddes}\,(\sigma)},$$

$$P_n(\alpha_1, \alpha_2, \alpha_3) = \sum_{\sigma \in \mathcal{Q}_n} \alpha_1^{\mathrm{dav}(\sigma)} \alpha_2^{\mathrm{dpa}(\sigma)} \alpha_3^{\mathrm{vdd}(\sigma)},$$

$$E_n(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = \sum_{\sigma \in \mathcal{Q}_n} \beta_1^{\mathrm{dplat}(\sigma)} \beta_2^{\mathrm{dasc}\,(\sigma)} \beta_3^{\mathrm{dd}(\sigma)} \beta_4^{\mathrm{uu}(\sigma)} \beta_5^{\mathrm{ddes}\,(\sigma)} \beta_6^{\mathrm{pasc}(\sigma)},$$

$$\alpha(\sigma) = \alpha_1^{\mathrm{dav}(\sigma)} \alpha_2^{\mathrm{dpa}(\sigma)} \alpha_3^{\mathrm{vdd}(\sigma)},$$

$$\beta(\sigma) = \beta_1^{\mathrm{dplat}(\sigma)} \beta_2^{\mathrm{dasc}\,(\sigma)} \beta_3^{\mathrm{dd}(\sigma)} \beta_4^{\mathrm{uu}(\sigma)} \beta_5^{\mathrm{ddes}\,(\sigma)} \beta_6^{\mathrm{pasc}(\sigma)}.$$

Let F_n denote the following seventeen-variable polynomials:

$$F_n := \sum_{\sigma \in \mathcal{Q}_n} \alpha(\sigma)\beta(\sigma) x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)} z^{\mathrm{des}\,(\sigma)} p^{\mathrm{lap}\,(\sigma)} q^{\mathrm{eud}(\sigma)} r^{\mathrm{rpd}(\sigma)} s^{\mathrm{apd}(\sigma)} t^{\mathrm{vv}(\sigma)}.$$

We can now present a generalization of Theorem 13.

Theorem 15. *For any $n \geqslant 1$, the seventeen-variable polynomial F_n has the expansion formula:*

$$F_n = t^n \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} \left(\frac{\delta_2}{t}\right)^i \left(\frac{\delta_1}{t}\right)^j \left(\frac{\delta}{t}\right)^k, \tag{22}$$

where $\delta = xyzpqrs$, $\delta_1 = \beta_4\beta_6 xyp + \beta_2\beta_5 xzq + \beta_1\beta_3 yzr$ and $\delta_2 = \alpha_1\beta_2\beta_4 x + \alpha_2\beta_1\beta_6 y + \alpha_3\beta_3\beta_5 z$.

A special case of (22) is given as follows.

Corollary 16. *The trivariate polynomial $M_n(\beta_1, \beta_4, \beta_5)$ is e-positive. More precisely,*

$$M_n(\beta_1, \beta_4, \beta_5) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k}(\beta_1 + \beta_4 + \beta_5)^{i+j}.$$

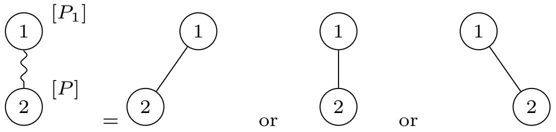

Fig. 2. Simplified ternary increasing trees, $Q_2 = xyzpqrs(xyp + xzq + yzr) = PP_1$.

As a unified extension of $N_n(p, q, r)$ and $P_n(\alpha_1, \alpha_2, \alpha_3)$, consider the six-variable polynomials

$$NP_n(p, q, r, \alpha_1, \alpha_2, \alpha_3) = \sum_{\sigma \in \mathcal{Q}_n} p^{\mathrm{lap}\,(\sigma)} q^{\mathrm{eud}(\sigma)} r^{\mathrm{rpd}(\sigma)} \alpha_1^{\mathrm{dav}(\sigma)} \alpha_2^{\mathrm{dpa}(\sigma)} \alpha_3^{\mathrm{vdd}(\sigma)}.$$

Corollary 17. *The six-variable polynomials $NP_n(p, q, r, \alpha_1, \alpha_2, \alpha_3)$ can be expanded as follows:*

$$NP_n(p, q, r, \alpha_1, \alpha_2, \alpha_3) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} (\alpha_1 + \alpha_2 + \alpha_3)^i (p + q + r)^j (pqr)^k.$$

When $p = q = r = 1$, we see that the polynomials $P_n(\alpha_1, \alpha_2, \alpha_3)$ are e-positive, i.e.,

$$P_n(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} 3^j (\alpha_1 + \alpha_2 + \alpha_3)^i.$$

Note that

$$E_n(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} \left(\beta_2\beta_4 + \beta_1\beta_6 + \beta_3\beta_5\right)^i \left(\beta_4\beta_6 + \beta_2\beta_5 + \beta_1\beta_3\right)^j.$$

Corollary 18. *We have $E_n(x, y, 1, 1, 1, 1) = E_n(1, 1, x, y, 1, 1) = E_n(1, 1, 1, 1, x, y)$ and the polynomials $E_n(x, y, 1, 1, 1, 1)$ are e-positive.*

3.3. Proof of Theorem 13

A *simplified ternary increasing tree* is a ternary increasing tree with no exterior nodes. In Fig. 2, we list the simplified ternary increasing trees of order 2, where the left figure represents the three different figures in the right. The *degree* of a vertex in a simplified ternary increasing tree is meant to be the number of its children.

The weight W_1 of $\sigma \in \mathcal{Q}_n$ is defined by

$$W_1(\sigma) = x^{\mathrm{asc}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)} z^{\mathrm{des}\,(\sigma)} p^{\mathrm{lap}\,\sigma)} q^{\mathrm{eud}(\sigma)} r^{\mathrm{rpd}(\sigma)} s^{\mathrm{apd}(\sigma)} t^{\mathrm{vv}(\sigma)}.$$

Using Table 1, one can get the corresponding weight of $C_n \in \mathrm{CQ}_n$, and we use $W_2(C_n)$ to denote it, see (27) for a more general case. In other words, if $\Gamma(\sigma) = C_n$, then

$W_1(\sigma) = W_2(C_n)$. In the following, we always set $P := xyzpqrs$, $P_1 := xyp + xzq + yzr$ and $P_2 := x + y + z$.

It is clear that $W_1(11) = W_2((0,0)) = xyzpqrs = P$. When $n = 2$, the weights of elements in $\mathcal{Q}_2$ and CQ_2 can be listed as follows:

$$\underbrace{2211 \leftrightarrow (0,0)(1,1)}_{xy^2z^2pqr^2s=Pyzr},\ \underbrace{1221 \leftrightarrow (0,0)(1,2)}_{x^2yz^2pq^2rs=Pxzq},\ \underbrace{1122 \leftrightarrow (0,0)(1,3)}_{x^2y^2zp^2qrs=Pxyp},$$

and the sum of weights is given by $P(xyp + xzq + yzr) = PP_1$.

Given $C_n = (0,0)(a_1,b_1)(a_2,b_2)\cdots(a_{n-1},b_{n-1}) \in \mathrm{CQ}_n$. Consider the elements in CQ_{n+1} generated from C_n by appending the 2-tuples (a_n, b_n), where $1 \leqslant a_n \leqslant n$ and $1 \leqslant b_n \leqslant 3$. Let T be the corresponding simplified ternary increasing tree of C_n. We can add $n+1$ to T as a child of a vertex, which is not of degree three. Let T' be the resulting simplified ternary increasing tree. We first give a labeling of T as follows. Label a leaf of T by P, a degree one vertex by P_1, a degree two vertex by P_2 and a degree three vertex by t. It is clear that the contribution of any leaf to the weight is $xyzpqrs$, so we set $P = xyzpqrs$.

The 2-tuples (a_n, b_n) can be divided into three classes:

- If $a_n \neq a_i$ for all $1 \leqslant i \leqslant n-1$, then we must add $n+1$ to a leaf of T. This operation corresponds to the following change of weights:

$$W_2(C_n) \to W_2(C_{n+1}) = W_2(C_n)(xyp + xzq + yzr), \tag{23}$$

  which yields the substitution $P \to PP_1$. The contribution of any leaf to the weight is $xyzpqrs$ and that of a degree one vertex is $xyp+xzq+yzr$ (which represents that this vertex may have a left child, a middle child or a right child). When we compute the corresponding weights of Stirling permutations, it follows from (23) that we need to set $P_1 = xyp + xzq + yzr$.
- If there is exactly one 2-tuple (a_i, b_i) in C_n such that $a_n = a_i$, then we must add $n+1$ to T as a child of the node a_i. Note that the node a_i already has one child, and $n+1$ becomes the second child of a_i. There are six cases to add $n+1$. This operation corresponds to the substitution $P_1 \to 2PP_2$. Since we have six cases to insert $n+1$ and the sum of increased weights is $2(x + y + z)$, so we set $P_2 = x + y + z$.
- If there are exactly two 2-tuples (a_i, b_i) and (a_j, b_j) in C_n such that $a_n = a_i = a_j$ and $i < j$, then we must add $n+1$ to T as the third child of a_i, and $n+1$ becomes a leaf with label P. This operation corresponds to the substitution $P_2 \to 3tP$.

The aforementioned three cases exhaust all the possibilities to construct SP-codes of length $n+1$ from a SP-code of length n by appending 2-tuples (a_n, b_n). In conclusion, each case corresponds to an application of the substitution rules defined by the following grammar:

$$I = \{P \to PP_1,\ P_1 \to 2PP_2,\ P_2 \to 3tP\}. \tag{24}$$

Note that the sum of degrees of all vertices in a simplified ternary increasing tree in $\mathcal{T}_n$ is n. Setting $w = P$, $v = P_1$ and $u = P_2$, it follows from (19) that

$$Q_n = D_G^n(x) = D_I^{n-1}(P) = t^n \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} \left(\frac{P_2}{t}\right)^i \left(\frac{P_1}{t}\right)^j \left(\frac{P}{t}\right)^k. \tag{25}$$

Upon taking $P = xyzpqrs,\ P_1 = xyp + xzq + yzr$ and $P_2 = x + y + z$, we arrive at Theorem 13. $\square$.

3.4. Proof of Theorem 14

Recall that $Q_n := Q_n(x, y, z, p, q, r, s, t)$. Let

$$R_n(P, P_1, P_2, t) = t^n \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} \left(\frac{P_2}{t}\right)^i \left(\frac{P_1}{t}\right)^j \left(\frac{P}{t}\right)^k.$$

Note that $\deg(P) + \deg(P_1) + \deg(P_2) + \deg(t) = n$ in any term of $R_n(P, P_1, P_2, t)$. Set

$$R_n(P, P_1, P_2, t) = P_2^n \widetilde{R}_n\left(\frac{P}{P_2}, \frac{P_1}{P_2}, \frac{t}{P_2}\right).$$

Upon taking $P = xyzpqrs,\ P_1 = xyp+xzq+yzr$ and $P_2 = x+y+z$, it follows from (25) that

$$Q_n = R_n(P, P_1, P_2, t) = (x+y+z)^n \widetilde{R}_n\left(\frac{xyzpqrs}{x+y+z}, \frac{xyp+xzq+yzr}{x+y+z}, \frac{t}{x+y+z}\right). \tag{26}$$

Note that

$$Q_n(1,1,1,p,q,r,s,t) = R_n(pqrs, p+q+r, 3, t) = 3^n \widetilde{R}_n\left(\frac{pqrs}{3}, \frac{p+q+r}{3}, \frac{t}{3}\right).$$

Substituting

$$p \to \frac{3xyp}{x+y+z},\ q \to \frac{3xzq}{x+y+z},\ r \to \frac{3yzr}{x+y+z},\ s \to \frac{s(x+y+z)^2}{9xyz},\ t \to \frac{3t}{x+y+z},$$

we find that

$$\frac{1}{3}pqrs \to \frac{xyzpqrs}{x+y+z},\ \frac{p+q+r}{3} \to \frac{xyp+xzq+yzr}{x+y+z},\ \frac{1}{3}t \to \frac{t}{x+y+z}.$$

It follows from (26) that

$$Q_n = \left(\frac{x+y+z}{3}\right)^n \times Q_n\left(1,1,1,\frac{3xyp}{x+y+z},\frac{3xzq}{x+y+z},\frac{3yzr}{x+y+z},\frac{s(x+y+z)^2}{9xyz},\frac{3t}{x+y+z}\right),$$

as desired. Since $C_n(x,y,z) = Q_n(x,y,z,1,1,1,1,1)$, it follows that

$$C_n(x,y,z) = \left(\frac{x+y+z}{3}\right)^n \times Q_n\left(1,1,1,\frac{3xy}{x+y+z},\frac{3xz}{x+y+z},\frac{3yz}{x+y+z},\frac{(x+y+z)^2}{9xyz},\frac{3}{x+y+z}\right),$$

which yields (21). This completes the proof. $\square$.

3.5. Proof of Theorem 15

Given a SP-code $C_n = ((0,0),(a_1,b_1),(a_2,b_2)\ldots,(a_{n-1},b_{n-1}))$. We make the symbols:

$$\begin{aligned}
\boxed{\mathrm{j}} &= \#\{a_i \mid (a_i,j) \notin C_n\},\\
\boxed{j_1,j_2} &= \#\{a_i \mid (a_i,j_1) \notin C_n \ \&\ (a_i,j_2) \notin C_n\},\\
\boxed{j_1,j_2,j_3} &= \#\{a_i \mid (a_i,j_1) \notin C_n \ \&\ (a_i,j_2) \notin C_n \ \&\ (a_i,j_3) \notin C_n\},\\
\boxed{\boxed{\mathrm{j}}} &= \#\{a_i \mid (a_i,j) \in C_n\},\\
\boxed{\boxed{j_1,j_2}} &= \#\{a_i \mid (a_i,j_1) \in C_n \ \&\ (a_i,j_2) \in C_n\},\\
\boxed{\boxed{j_1,j_2,j_3}} &= \#\{a_i \mid (a_i,j_1) \in C_n \ \&\ (a_i,j_2) \in C_n \ \&\ (a_i,j_3) \in C_n\},\\
\boxed{j_1}\boxed{\boxed{j_2}} &= \#\{a_i \mid (a_i,j_1) \notin C_n \ \&\ (a_i,j_2) \in C_n\},\\
\boxed{j_1}\boxed{\boxed{j_2,j_3}} &= \#\{a_i \mid (a_i,j_1) \notin C_n \ \&\ (a_i,j_2) \in C_n \ \&\ (a_i,j_3) \in C_n\}.
\end{aligned}$$

Recall that

$$\alpha(\sigma) = \alpha_1^{\mathrm{dav}(\sigma)}\alpha_2^{\mathrm{dpa}(\sigma)}\alpha_3^{\mathrm{vdd}(\sigma)},\ \beta(\sigma) = \beta_1^{\mathrm{dplat}(\sigma)}\beta_2^{\mathrm{dasc}\,(\sigma)}\beta_3^{\mathrm{dd}(\sigma)}\beta_4^{\mathrm{uu}(\sigma)}\beta_5^{\mathrm{ddes}\,(\sigma)}\beta_6^{\mathrm{pasc}(\sigma)}.$$

The weight W_3 of $\sigma \in \mathcal{Q}_n$ is defined by

$$W_3 := \alpha(\sigma)\beta(\sigma)x^{\mathrm{asc}\,(\sigma)}y^{\mathrm{plat}\,(\sigma)}z^{\mathrm{des}\,(\sigma)}p^{\mathrm{lap}\,(\sigma)}q^{\mathrm{eud}(\sigma)}r^{\mathrm{rpd}(\sigma)}s^{\mathrm{apd}(\sigma)}t^{\mathrm{vv}(\sigma)}.$$

It follows from Table 1 that the corresponding weight W_4 of $C_n \in \mathrm{CQ}_n$ is given as follows:

$$W_4 := \alpha(C_n)\beta(C_n)x^{\boxed{1}}y^{\boxed{2}}z^{\boxed{3}}p^{\boxed{1,2}}q^{\boxed{1,3}}r^{\boxed{2,3}}s^{\boxed{1,2,3}}t^{\boxed{\boxed{1,2,3}}}. \tag{27}$$

where

$$\alpha(C_n) = \alpha_1^{\boxed{1}\boxed{\boxed{2,3}}}\alpha_2^{\boxed{2}\boxed{\boxed{1,3}}}\alpha_3^{\boxed{3}\boxed{\boxed{1,2}}},$$
$$\beta(C_n) = \beta_1^{\boxed{2}\boxed{\boxed{1}}}\beta_2^{\boxed{1}\boxed{\boxed{2}}}\beta_3^{\boxed{3}\boxed{\boxed{1}}}\beta_4^{\boxed{1}\boxed{\boxed{3}}}\beta_5^{\boxed{3}\boxed{\boxed{2}}}\beta_6^{\boxed{2}\boxed{\boxed{3}}}.$$

Let T be the corresponding simplified ternary increasing tree of C_n. A labeling of T as follows. Label a leaf of T by δ, a degree one vertex by δ_1, a degree two vertex by δ_2 and a degree three vertex by t. Consider all the possibilities to construct SP-codes of length $n+1$ from a SP-code of length n by appending 2-tuples (a_n, b_n). In the same way as in the proof of Theorem 13, each case corresponds to an application of the substitution rules defined by the following grammar:

$$J = \{\delta \to \delta\delta_1,\ \delta_1 \to 2\delta\delta_2,\ \delta_2 \to 3t\delta\},$$

where $\delta = xyzpqrs,\ \delta_1 = \beta_4\beta_6 xyp + \beta_2\beta_5 xzq + \beta_1\beta_3 yzr$ and $\delta_2 = \alpha_1\beta_2\beta_4 x + \alpha_2\beta_1\beta_6 y + \alpha_3\beta_3\beta_5 z$. Setting $w = \delta$, $v = \delta_1$ and $u = \delta_2$, as in (25), we get

$$F_n = D_J^{n-1}(\delta) = t^n \sum_{i+2j+3k=2n+1} \gamma_{n,i,j,k} \left(\frac{\delta_2}{t}\right)^i \left(\frac{\delta_1}{t}\right)^j \left(\frac{\delta}{t}\right)^k.$$

Then upon taking $\delta = xyzpqrs,\ \delta_1 = \beta_4\beta_6 xyp + \beta_2\beta_5 xzq + \beta_1\beta_3 yzr$ and $\delta_2 = \alpha_1\beta_2\beta_4 x + \alpha_2\beta_1\beta_6 y + \alpha_3\beta_3\beta_5 z$, we arrive at (22), as desired. This completes the proof. □.

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of the manuscript entitled "Stirling permutation codes. II".

Acknowledgments

We wish to thank the referees for invaluable comments and suggestions. The first author was supported by the National Natural Science Foundation of China (Grant number 12071063) and Taishan Scholars Program of Shandong Province (No. tsqn202211146). The third author was supported by the National Science and Technology Council (Grant number: MOST 113-2115-M-017-005-MY2).

Data availability

No data was used for the research described in the article.

References

[1] R.M. Adin, F. Brenti, Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 27 (2001) 210–224.
[2] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math. 23 (2008/2009) 401–406.
[3] P. Brändén, Actions on permutations and unimodality of descent polynomials, Eur. J. Comb. 29 (2008) 514–531.
[4] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, Eur. J. Comb. 15 (1994) 417–441.
[5] R.A. Brualdi, Stirling pairs of permutations, Graphs Comb. 36 (2020) 1145–1162.
[6] J.D. Buckholtz, Concerning an approximation of Copson, Proc. Am. Math. Soc. 14 (1963) 564–568.
[7] L. Carlitz, The coefficients in an asymptotic expansion, Proc. Am. Math. Soc. 16 (1965) 248–252.
[8] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theor. Comput. Sci. 117 (1993) 113–129.
[9] W.Y.C. Chen, A.M. Fu, A context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345 (1) (2022) 112661.
[10] W.Y.C. Chen, R.X.J. Hao, H.R.L. Yang, Context-free grammars and multivariate stable polynomials over Stirling permutations, in: V. Pillwein, C. Schneider (Eds.), Algorithmic Combinatorics: Enumerative Combinatorics, Special Functions and Computer Algebra, Springer, 2021, pp. 109–135.
[11] W.Y.C. Chen, A.M. Fu, S.H.F. Yan, The Gessel correspondence and the partial γ-positivity of the Eulerian polynomials on multiset Stirling permutations, Eur. J. Comb. 109 (2023) 103655.
[12] D. Dumont, Une généralisation trivariée symétrique des nombres eulériens, J. Comb. Theory, Ser. A 28 (1980) 307–320.
[13] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sémin. Lothar. Comb. 37 (1996) B37a.
[14] S. Elizalde, Descents on quasi-Stirling permutations, J. Comb. Theory, Ser. A 180 (2021) 105429.
[15] D. Foata, M.P. Schützenberger, Théorie Géometrique des Polynômes Eulériens, Lecture Notes in Math., vol. 138, Springer, Berlin, 1970.
[16] J. Garloff, D.G. Wagner, Hadamard products of stable polynomials are stable, J. Math. Anal. Appl. 202 (1996) 797–809.
[17] I. Gessel, R.P. Stanley, Stirling polynomials, J. Comb. Theory, Ser. A 24 (1978) 25–33.
[18] I. Gessel, Y. Zhuang, Plethystic formulas for permutation enumeration, Adv. Math. 375 (2) (2020) 107370.
[19] J. Haglund, M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, Eur. J. Comb. 33 (2012) 477–487.
[20] S. Janson, Plane recursive trees, Stirling permutations and an urn model, in: Proceedings of Fifth Colloquium on Mathematics and Computer Science, Discrete Math. Theor. Comput. Sci. AI (2008) 541–547.
[21] S. Janson, M. Kuba, A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, J. Comb. Theory, Ser. A 118 (1) (2011) 94–114.
[22] K.Q. Ji, Z. Lin, The binomial-Stirling-Eulerian polynomials, Eur. J. Comb. 120 (2024) 103962.
[23] M. Kuba, A.L. Varvak, On path diagrams and Stirling permutations, Sémin. Lothar. Comb. 82 (2021) B82c.
[24] Z. Lin, J. Ma, P.B. Zhang, Statistics on multipermutations and partial γ-positivity, J. Comb. Theory, Ser. A 183 (2021) 105488.
[25] S.H. Liu, MacMahon's equidistribution theorem for k-Stirling permutations, Adv. Appl. Math. 128 (2021) 102193.
[26] J. Ma, S.-M. Ma, Y.-N. Yeh, Recurrence relations for binomial-Eulerian polynomials, arXiv:1711.09016v2.
[27] S.-M. Ma, Y. Wang, q-Eulerian polynomials and polynomials with only real zeros, Electron. J. Comb. 15 (2008), #R17.
[28] S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, Electron. J. Comb. 24 (4) (2017), #P4.27.

[29] S.-M. Ma, J. Ma, Y.-N. Yeh, The ascent-plateau statistics on Stirling permutations, Electron. J. Comb. 26 (2) (2019), #P2.5.

[30] S.-M. Ma, J. Ma, Y.-N. Yeh, γ-positivity and partial γ-positivity of descent-type polynomials, J. Comb. Theory, Ser. A 167 (2019) 257–293.

[31] S.-M. Ma, J. Ma, J. Yeh, Y.-N. Yeh, Eulerian pairs and Eulerian recurrence systems, Discrete Math. 345 (3) (2022) 112716.

[32] S.-M. Ma, H. Qi, J. Yeh, Y.-N. Yeh, Stirling permutation codes, J. Comb. Theory, Ser. A 199 (2023) 105777.

[33] T.K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2) (2007) 561–610.

[34] T.K. Petersen, Eulerian Numbers, Birkhäuser/Springer, New York, 2015.

[35] S. Ramanujan, Collected Papers, Cambridge, 1927, p. 26.

[36] C.D. Savage, G. Viswanathan, The $1/k$-Eulerian polynomials, Electron. J. Comb. 19 (2012), #P9.

[37] C.D. Savage, M. Visontai, The s-Eulerian polynomials have only real roots, Trans. Am. Math. Soc. 367 (2015) 763–788.

[38] R.P. Stanley, Longest alternating subsequences of permutations, Mich. Math. J. 57 (2008) 675–687.

[39] J.R. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, Adv. Math. 106 (1994) 244–301.

[40] Y. Zhuang, Counting permutations by runs, J. Comb. Theory, Ser. A 142 (2016) 147–176.

[41] Y. Zhuang, Eulerian polynomials and descent statistics, Adv. Appl. Math. 90 (2017) 86–144.